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Surface displacements due to pressure modifications induced by tunnels

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ABSTRACT: Tunnel excavation in an aquifer rock mass modifies water pressure which generates deformations and surface displacements. The amplitude and shape of surface displacements are related to a large number of parameters which characterise the rock mass, the aquifer, the tunnel and their interactions. A weak procedure is used in this paper to simplify the basic equations. The simplified equations serve to predict surface displacement behaviour for varying anisotropy ratios, rock mass dimensions and tunnel properties.

1 INTRODUCTION

Tunnel excavation produces different kinds of ground movement whose amplitude depends on the generating effect and its location. The amplitude of surface displacements which are produced by volume loss, such as tunnel convergence or opening collapse, decrease with their depth location. Surface displacements which are produced by pressure modifications increase with tunnel depth. From the different generating effects of surface displacements, those which are due to water may become predominant for tunnels excavated at great depth. Pressure modifications bring about subsidence and horizontal displacements which are of primary importance for the monitoring instances of surface and underground structures. Certain structures, such as arch dams, are sensitive to horizontal displacements which may widen or narrow the valleys they are closing. In urban areas, both vertical and horizontal deformations are needed to assess the potential exposure risk to buildings from damage charts.

Many concepts are involved when simulating ground displacements induced by pressure modifications. Some of these are the space delimitation of the phenomena, the boundary behaviour and the rock mass properties. This paper partially tackles these concepts. It will attempt to clarify the parameters role needed for these concepts and find their relation to surface displacements based on a simplified set of equations.

The basic equations can be simplified using numerical techniques or physical observations. In this paper a numerical technique is used. A partial application of the finite element method is used to eliminate one space dimension from the basic equations. This is called a weak approximation and the resulting equations are called semi discrete. This technique is applied considering differing behaviour of the rock mass base, which may slide or be rigid. For each behaviour, a set of semi discrete equations is obtained and may serve in evaluating different effects on the amplitude and shape of surface displacements, such as the rock mass dimensions, anisotropy ratios and the tunnel depth and geometry.

2 BASIC EQUATIONS

Induced pressures and displacements which are brought about by an excavated tunnel may be classed, in most cases, as 2D phenomena. This is a highly limiting hypothesis to which the following is added; the pressures brought about by the tunnel excavation are known. They are, in the case of linear behaviour, independent of displacements and may be calculated separately. Displacements follow pressure modifications instantaneously when inertia and viscosity are ignored. This will simplify the basic equations and enables the accent to be put on the 2D induced displacements.

The basic equations are the equilibrium equations

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau}{\partial z} = 0 \qquad [2.1]$$

$$\frac{\partial \tau}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \qquad [2.2]$$

and the elastic relationships coupled to the Fillunger-Terzaghi porous material effective stress or the Dalton-Truesdell mixture partial stress (zero order flux stress)

$$\sigma_y = E_y(\varepsilon_y + \lambda_{yz}\varepsilon_z) - p \qquad [2.3]$$

$$\sigma_z = E_z(\varepsilon_z + \lambda_{zy}\varepsilon_y) - p \qquad [2.4]$$

$$\tau = 2G_{yz}\varepsilon_{yz} \qquad [2.5]$$

where σ_y, σ_z and τ are the horizontal, vertical and shear components of the total stress tensor, ε_y, ε_z and ε_{yz} are the horizontal, vertical and shear components of the strain tensor, p is water pressure, E_y, E_z and G_{yz} are the elastic and shear modulus which are positive and λ_{yz} and λ_{zy} are crossed effect coefficients which verify $E_y\lambda_{yz}$=$E_z\lambda_{zy}$ and $\lambda_{yz}\lambda_{zy} \leq 1$.

The elastic coefficients E_y, E_z, λ_{yz}, λ_{zy} and G_{yz} are those of a 2D orthotropic material which fit major elastic continuous models for rocks and soils. Many models are proposed for a direct evaluation of the elastic coefficients for jointed rocks or porous soils. These models describe a gross response behaviour of a representative element containing a large number of joints for a rock or a large number of void inclusions for a soil. Here it is supposed that in the case of a rock mass there are at least two intersecting sets of fissures.

Generally, measured or computed elastic coefficients are those of 3D elasticity and a conversion is necessary for 2D elasticity. The last five columns of Table 1 give the 2D elastic coefficients in terms of 3D cubic and tetragonal coefficients for both deformation modes, which are called plane deformation and plane stress.

Strain definitions complete the set of equations and relations [2.1] to [2.5] and are

$$\varepsilon_y = \frac{\partial v}{\partial y} \quad \varepsilon_z = \frac{\partial w}{\partial z} \quad 2\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \qquad [2.6]$$

where v and w are the horizontal and vertical displacements. oy and oz are the horizontal and vertical axes.

3 THE TUNNEL VERTICAL AXIS

The tunnel vertical axis has special properties. In the low shear model, it is the location of the inflexion point for the horizontal displacement, the location of the pressure modification maximum value and the location of the vertical displacements maximum value. It will become a sliding vertical axis if it is an axis of symmetry. The tunnel vertical axis behaves as a boundary even if it is not a material one. It is natural to adopt it as such since its position is indisputable. The choice of the position of the other vertical and horizontal boundaries of the rock mass remains. These should result from geological surveys. However, if their positions are unknown, they must figure as parameters during the entire analyses of the strain effects on surface structures.

Table 1. Relations between 3D elastic coefficients and 2D elastic coefficients.

Material	Coefficients*	Mode	E_y	E_z	λ_{yz}	λ_{zy}	G_{yz}
Cubic**	E ν G	Plane deformation	$\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}$	$\frac{(1-\nu)E}{(1+\nu)(1-2\nu)}$	$\frac{\nu}{1-\nu}$	$\frac{\nu}{1-\nu}$	G
		Plane stress	$\frac{E}{(1+\nu)(1-\nu)}$	$\frac{E}{(1+\nu)(1-\nu)}$	ν	ν	G
Tetragonal***	E ν G E' ν' G'	Plane deformation	$\frac{(E'-\nu'^2 E)E}{(1+\nu)[(1-\nu)E'-2\nu'^2 E]}$	$\frac{(1-\nu)E'^2}{(1-\nu)E'-2\nu'^2 E}$	$\frac{\nu'(1+\nu)E'}{E'-\nu'^2 E}$	$\frac{\nu' E}{(1-\nu)E'}$	G'
		Plane stress	$\frac{EE'}{E'-\nu'^2 E}$	$\frac{E'^2}{E'-\nu'^2 E}$	ν'	$\frac{\nu' E}{E'}$	G'

* 3D elastic coefficients of an orthotropic material are the elastic modulus: E_1,E_2,E_3, Poisson coefficients: ν_{12},ν_{13},ν_{23} and shear modulus: G_{12},G_{13},G_{23}.

** Cubic material is a special orthotropic material which has three independant coefficients defined by: E_1=E_2=E_3≡E, ν_{12}=ν_{13}=ν_{23}≡ν, G_{12}=G_{13}=G_{23}≡G. Isotopic elasticity is a special cubic material for wich G=E/2(1+ν).

*** Tetragonal material: E_1=E_2≡E, E_3≡E', ν_{12}≡ν, ν_{13}=ν_{23}≡ν',G_{12}≡G, G_{13}=G_{23}≡G'. Transverse isotropy is a special case for which G=E/2(1+ν). Saint Venant material is a special case for which 1/G'=(1+ν)/E+(1+ν')/E'.

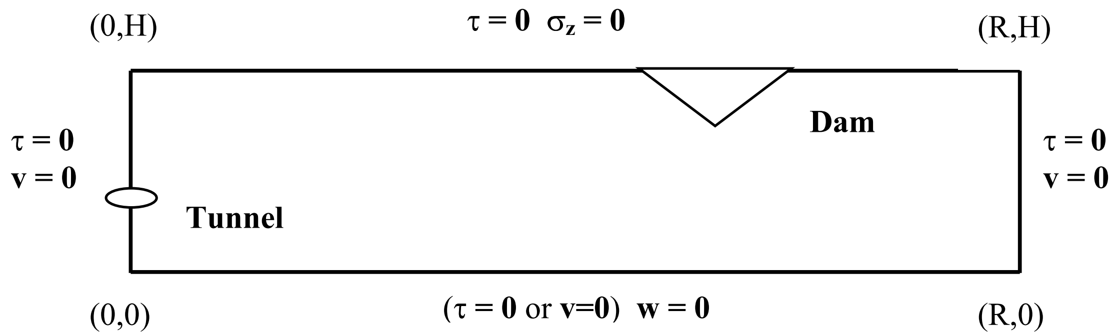

Figure 1. Schematic representation of the rock mass with a tunnel and a dam.

All of the preceding comments suggest the consideration of a rock mass bordered by the vertical tunnel axis from the beginning. The tunnel vertical axis will also be considered as an axis of symmetry to allow the horizontal displacement to be zero there. The rock mass and boundary conditions are shown in Figure 1. The upper surface is unloaded. Two conditions are possible at the base:

- The base may slide on its support and in this case the shear stress is zero.
- The base cannot move horizontally and the horizontal displacement is zero there.

These antagonist conditions represent the limiting cases of possible behaviour of the base and they lead to two different evolutions of the mass. They will be differentiated and called the "sliding base" and the "rigid base". Concerning the final and right boundary, sliding conditions are considered.

4 RIGID BASE MASS

The rigid base mass does not lend itself to a global analysis of an analytical character. A detailed analysis would be possible with the aid of a bi-harmonic analysis or using the integral method with Green functions. These are two possibilities that would be interesting to explore but for the moment it is through a semi-weak approach that the rigid base model yields some useful information. The semi-weak method is a partial application of the finite element method which reduces the equations to new, simplified and approximate equations. This method is described in the Appendix and is applied to both mass models: the rigid and sliding base.

4.1 *The rigid base mass in the weak formulation*

The simplified equations that are obtained by applying the weak formulation to the rigid base model are Equations [A9] and [A10] in the Appendix. They only take into account surface displacements. It is, therefore, possible to use the following notations without risk of confusion:

- v(y) for v(y,H) which is the horizontal surface displacement value.
- w(y) for w(y,H) which is the vertical surface displacement value.

The simplified equations of the rigid base mass are coupled but, luckily, they separate when the crossed effect coefficient is such that $E_y\lambda_{yz}$ is equal to G_{yz} and, in this case, they assume the following forms:

$$\frac{d^2v}{dy^2}-\frac{3G_{yz}}{E_y}\frac{v}{H^2}=\frac{3}{HE_y}\int_0^H\frac{z}{H}\frac{\partial p}{\partial y}dz \qquad [4.1]$$

$$\frac{d^2w}{dy^2}-\frac{3E_z}{G_{yz}}\frac{w}{H^2}=\frac{3}{HG_{yz}}\int_0^H\frac{z}{H}\frac{\partial p}{\partial z}dz \qquad [4.2]$$

4.2 *Uncoupled steady state displacements*

Final pressure can be expressed in an integral form, which does not really help to extract surface displacements from [4.1] and [4.2] in a simple parametric form. Remembering that pressure is a harmonic function, the task becomes easier by decomposing pressure into a sum of harmonic functions of the form $e^{-\beta y}\cos(\alpha z)$; $\alpha/\beta=(k_y/k_z)^{1/2}$ and k_y and k_z are the horizontal and vertical permeabilities. There is still a simpler form, which is the weak form; a semi weak pressure may be obtained by applying the same formalism described in the Appendix to the pressure governing equation. The semi discrete triangular shaped pressure in an infinite strip (R is infinite in Figure 1) of height H is

$$p(y,z)=\Delta p(1-\frac{z}{H})e^{-\sqrt{\frac{3k_z}{k_y}}\frac{y}{H}} \qquad [4.3]$$

in which Δp is determined from pressure values. It will be chosen such that [4.3] is closest to pressure modifications at the tunnel vertical axis. For a rapid and practical estimation of Δp, both sides of [4.3] are integrated considering H as twice the tunnel depth and p(y,z) is replaced by its integral approximation. This leads to

$$\Delta p = \frac{3\gamma \ln 3}{2\pi} \frac{Q}{\sqrt{k_y k_z}} \qquad [4.4]$$

Q is the volume of water which flows into the tunnel per unit time and unit tunnel length and is related to k_y and k_z and to h_r, h and r being the resultant head, tunnel depth and radius. γ is the specific weight.

Substituting [4.3] into [4.1] and [4.2], uncoupled surface displacements induced by pressure modifications in an infinite strip are obtained as

$$v = \frac{H\Delta p}{2\sqrt{3}E_y} \sqrt{\frac{k_z}{k_y}} \frac{e^{-\sqrt{\frac{3k_z}{k_y}}\frac{y}{H}} - e^{-\sqrt{\frac{3G_{yz}}{E_y}}\frac{y}{H}}}{G_{yz}/E_y - k_z/k_y} \qquad [4.5]$$

$$w = \frac{H\Delta p}{2G_{yz}} \frac{e^{-\sqrt{\frac{3k_z}{k_y}}\frac{y}{H}} - \sqrt{k_z G_{yz}/k_y E_z}\, e^{-\sqrt{\frac{3E_z}{G_{yz}}}\frac{y}{H}}}{E_z/G_{yz} - k_z/k_y} \qquad [4.6]$$

Boundary conditions are v(0)= 0 and dw(0)/dy =0. The latter is the remainder of the condition of zero shear on the tunnel vertical axis when the horizontal displacement is constant or zero.

4.3 *Displacement properties*

Displacement signs are identical to the pressure modification. Δp is negative for an excavated tunnel. In this case surface points will settle and move horizontally towards the tunnel axis. If Δp is positive, surface points will heave and move outwards. The amplitude of the surface horizontal displacement increases from zero at the tunnel vertical axis to attain a maximum and decrease back to zero. The maximum horizontal displacement amplitude is obtained at

$$\frac{y_{v\_max}}{H} = \frac{1}{2\sqrt{3}} \frac{\ln \frac{k_z}{k_y}\frac{E_y}{G_{yz}}}{\sqrt{\frac{k_z}{k_y}} - \sqrt{\frac{G_{yz}}{E_y}}} \qquad [4.7]$$

Vertical surface displacement amplitude is maximum at the origin and decreases to zero moving outwards. Its inflexion point is attained at

$$\frac{y_{w\_inf\,lexion}}{H} = \frac{1}{2\sqrt{3}} \frac{\ln \frac{k_z}{k_y}\frac{G_{yz}}{E_z}}{\sqrt{\frac{k_z}{k_y}} - \sqrt{\frac{E_z}{G_{yz}}}} \qquad [4.8]$$

y_{v_max} and $y_{w\_inflexion}$ locations depend on the outer boundary which has been, for practical purposes, moved to infinity. They might be re-written for a bounded rock mass, possibly using a harmonic pressure decomposition. This is needed to refine the prediction concerning surface displacements and the position of their particular points.

As an example, if the surface structure is an arch dam and is located between 0 and y_{v_max}, its abutments will move toward each other. If the arch dam is located beyond y_{v_max}, the abutments will move in opposite directions. If y_{v_max} is in the middle of the dam, the latter will be shifted toward the tunnel axis but not deformed horizontally.

4.4 *Isotropic displacements*

The de-coupling relation $E_y\lambda_{yz} = G_{yz}$ in the case of isotropic behaviour is satisfied with a unique value of Poisson's coefficient in each of the deformation modes. ν is 1/4 for plane deformation and 1/3 for plane stress (Table 1). G_{yz} is noted G and E/G is $2(1+\nu)$; E is the 3D isotropic elastic modulus. Cross coefficients λ_{yz} and λ_{zy} are both equal to 1/3 and are independent of the deformation mode. Ratios G_{yz}/E_y, G_{yz}/E_z and k_z/k_y are equal to 1/3, 1/3 and 1, respectively. Inserting these values in [4.5] and [4.6] leads to surface displacements of an isotropic infinite rigid base strip. Their amplitudes are shown in Figure 2.

5 CONCLUSION

The semi discrete weak model has demonstrated its usefulness in isolating parameters of the water-ground-tunnel interaction and looking at their individual effects. An important point has not been treated; it is the precision and application range of the weak equations. Tunnel excavation consolidates the rock mass and then drains the aquifer, lowering the water table. Displacements in this paper reflect the consolidating part, since the water table has been implicitly assumed to hold jointly to the surface.

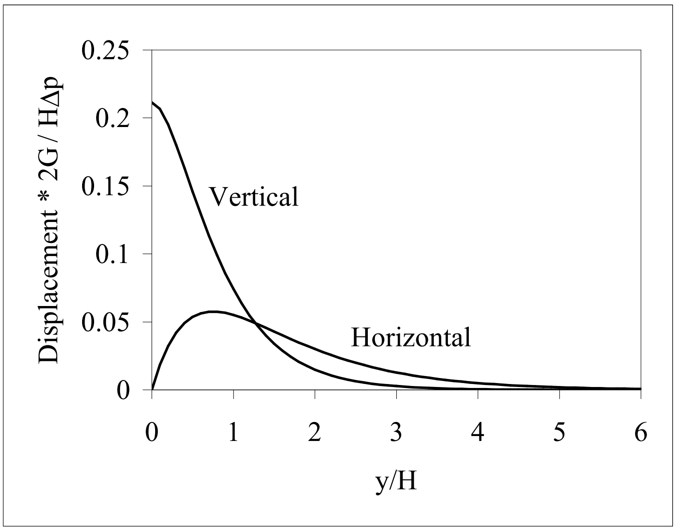

Figure 2. Amplitudes of uncoupled steady state surface displacements of an isotropic infinite rigid base strip.

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APPENDIX: THE SEMI-DISCRETE WEAK FORMULATION

The semi-weak formulation consists of a partial weighted integration of the basic equations. They are made discrete in one space direction and analytical in the other.

A1 *The semi weak procedure*

A set of nodal lines z_i and functions $\phi_i(z)$ are chosen with the following properties:

$$\phi_i(z_j) = \delta_{ij} \; ; \quad i,j \geq 1 \qquad \text{[A1]}$$

where δ_{ij} is Kronecker's delta. The functions ϕ_i are used as interpolation functions for the displacements:

$$v(y,z) = \sum_i \phi_i(z) v(y,z_i) \qquad \text{[A2]}$$

$$w(y,z) = \sum_i \phi_i(z) w(y,z_i) \qquad \text{[A3]}$$

The first component of the equilibrium equation [2.1] is multiplied by the functions $\phi_i(z)$ whenever $v(y,z_i)$ is non zero. The second component of the equilibrium equation [2.2] is multiplied by the functions $\phi_i(z)$ whenever $w(y,z_i)$ is non zero. The resulting products are integrated over a vertical line. Taking into account the conditions at the base and surface shown in Figure 1, the integration leads to

$$\int_0^H [\phi_i \frac{\partial \sigma_y}{\partial y} - \tau \frac{d\phi_i}{dz}] dz = 0 \; ; \quad i \geq 1 \text{ and } v(y,z_i) \neq 0 \qquad \text{[A4]}$$

$$\int_0^H [\phi_i \frac{\partial \tau}{\partial y} - \sigma_z \frac{d\phi_i}{dz}] dz = 0 \; ; \quad i \geq 1 \text{ and } w(y,z_i) \neq 0 \qquad \text{[A5]}$$

The final weak equations are obtained by inserting the interpolations [A2] and [A3] into strains [2.6] which are taken into the material relations [2.3] to [2.5] which are then inserted in [A4] and [A5].

A2 *Application : Sliding base mass*

The linear interpolation functions over the segment [0,H] are chosen. There are two functions which are $\phi_1(z)$ = 1-z/H and $\phi_2(z)$ = z/H. The nodal lines z_1 and z_2 are 0 and H. Only the horizontal and vertical displacements at the base and at the surface are involved. Thus, it is possible to use the following notations without risk of confusion:
- w(y) for w(y,H) which is the value of the vertical displacement at the surface.
- v(y) for (v(y,0)+v(y,H))/2 which is the mean value of the horizontal displacement.
- Δ(y) for v(y,H)-v(y,0) which is the difference between the surface displacements and those at the base.

Applying the procedure given in section A1, the following equations may be obtained after some manipulations:

$$\frac{d^2 v}{dy^2} + \frac{\Lambda}{HE_y}\frac{dw}{dy} = \frac{1}{HE_y}\int_0^H \frac{\partial p}{\partial y} dz \qquad \text{[A6]}$$

$$\frac{d^2 \Delta}{dy^2} - \frac{12G_{yz}}{E_y}\frac{\Delta}{H^2} - \frac{6G_{yz}}{E_y H}\frac{dw}{dy} = \frac{6}{HE_y}\int_0^H (\frac{2z}{H} - 1)\frac{\partial p}{\partial y} dz \qquad \text{[A7]}$$

$$\frac{d^2 w}{dy^2} - \frac{3E_z w}{G_{yz}H^2} + \frac{3}{2H}\frac{d\Delta}{dy} - \frac{3\Lambda}{G_{yz}H}\frac{dv}{dy} = \frac{3}{G_{yz}H}\int_0^H \frac{z}{H}\frac{\partial p}{\partial z} dz \qquad \text{[A8]}$$

where $\Lambda = E_y\lambda_{yz} = E_z\lambda_{zy}$. The boundary conditions are v(0)=v(R)=Δ(0)=Δ(R)=dw/dy(0)=dw/dy(R)=0.

A3 *Application : Rigid base mass*

The same nodal lines z_1 and z_2 and the same interpolation functions ϕ_1 and ϕ_2 from section A2 are used. Here, only v(y,H) and w(y,H) are involved. The following notation is used:
- v(y) for v(y,H) which is the surface horizontal displacement.
- w(y) for w(y,H) which is the surface vertical displacement.

Applying the procedure in A1 leads to the following equations:

$$\frac{d^2 v}{dy^2} - \frac{3G_{yz}}{E_y}\frac{v}{H^2} + \frac{3(\Lambda - G_{yz})}{2E_y H}\frac{dw}{dy} = \frac{3}{HE_y}\int_0^H \frac{z}{H}\frac{\partial p}{\partial y} dz \qquad \text{[A9]}$$

$$\frac{d^2 w}{dy^2} - \frac{3E_z}{G_{yz}}\frac{w}{H^2} + \frac{3(G_{yz} - \Lambda)}{2G_{yz}H}\frac{dv}{dy} = \frac{3}{HG_{yz}}\int_0^H \frac{z}{H}\frac{\partial p}{\partial z} dz \qquad \text{[A10]}$$

where $\Lambda = E_y\lambda_{yz} = E_z\lambda_{zy}$. The boundary conditions are v(0)=v(R)=dw/dy(0)= dw/dy(R)=0.